

Extended Decentralized Linear-Quadratic-Gaussian Control

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Abstract

A straightforward extension of a solution to the decentralized linear-Quadratic-Gaussian problem is proposed that allows its use for commonly encountered classes of problems that are currently solved with the extended Kalman filter. This extension allows the system to be partitioned in such a way as to exclude the nonlinearities from the essential algebraic relationships that allow the estimation and control to be optimally decentralized.

Introduction

A distributed system consisting of K nodes interconnected via a communications network could be controlled using a decentralized controller framework that operates in parallel over the network. For such problems, a solution that minimizes data transmission requirements, in the context of state estimation of Gauss-Markov systems for distributed processing of local data and their integration for constructing optimal global estimates, has been given by Speyer [1]. In [1], the decentralized estimator was placed in a linear-quadratic-Gaussian (LQG) control setting. Generalizations of [1] may be found in [2] and [3], and in [4], this work served as the basis for a fault-tolerant multi-sensor navigation architecture. In [5], the decentralized LQG control is extended to the decentralized linear-exponential-Gaussian control which is related to deterministic \mathcal{H}_∞ control synthesis.

As literally formulated in [1], the approach is valid for linear time-invariant systems only. As with the standard LQG problem, extension to linear time-varying systems requires that each node propagate its filter covariance forward and controller Riccati matrix backward at each time step. Furthermore, commonly used ad hoc techniques for problems with nonlinear state and output relations such as the extended Kalman filter (EKF) violate certain linearity assumptions inherent in the decentralization of the estimation process developed in [1].

The contribution of this paper is to extend the linear decentralized controller of [1] in a somewhat similar fashion to the extended Kalman filter. This extension allows the system to be partitioned in such a way as to exclude the nonlinearities from the essential algebraic relationships that allow the estimation and control to be optimally decentralized.

Problem Statement

It is desired to minimize over $U^j(t), j = 1, 2, \dots, K$,

$$J = \frac{1}{2} \mathbb{E} \left[\{X(t_N) - X^R(t_N)\}^\top S_N \{X(t_N) - X^R(t_N)\} + \int_{t_1}^{t_N} \left\{ [X(t) - X^R(t)]^\top W(t) [X(t) - X^R(t)] + \sum_{j=1}^K [U^j(t) - U^{Rj}(t)]^\top V^j(t) [U^j(t) - U^{Rj}(t)] \right\} dt \right] \quad (1)$$

subject to

$$\dot{X}(t) = f(X, t) + \sum_{j=1}^K b^j(U^j, t) + G(t)w(t); \quad X(t_1) = X_1 \quad (2)$$

$$\dot{X}^R(t) = f(X^R, t) + \sum_{j=1}^K b^j(U^{Rj}, t); \quad X^R(t_1) = X_1^R \quad (3)$$

$$Y_i^j = g^j(X, t_i) + v_i^j; \quad j = 1, 2, \dots, K \quad (4)$$

where

$$\mathbb{E}[\{X_1 - X_1^R\}] = 0; \quad \mathbb{E}[\{X_1 - X_1^R\}\{X_1 - X_1^R\}^\top] = P_1 \quad (5)$$

$$\mathbb{E}[w(t)] = 0; \quad \mathbb{E}[w(t)w(\tau)^\top] = Q(t)\delta(t - \tau) \quad (6)$$

$$\mathbb{E}[v_i^j] = 0; \quad \mathbb{E}[v_i^j\{v_h^j\}^\top] = R_i^j\delta_{ih}; \quad \mathbb{E}[v_i^j\{v_i^\ell\}^\top] = 0 \quad (7)$$

$$\mathbb{E}[\{X_1 - X_1^R\}w(t)^\top] = 0; \quad \mathbb{E}[w(t)\{v_i^j\}^\top] = 0$$

$$\mathbb{E}[\{X_1 - X_1^R\}\{v_i^j\}^\top] = 0 \quad (8)$$

In some applications, it may be desired to control only some linear combination of the state deviations from the reference,

$$z(t) = M[X(t) - X^R(t)] \quad (9)$$

In such a case,

$$\begin{aligned} \tilde{J} = & \frac{1}{2} \mathbb{E} \left[z(t_N)^\top \tilde{S}_N z(t_N) \right. \\ & \left. + \int_{t_1}^{t_N} \left\{ z(t)^\top \tilde{W}(t) z(t) + \sum_{j=1}^K [U^j(t) - U^{Rj}(t)]^\top V^j(t) [U^j(t) - U^{Rj}(t)] \right\} dt \right] \end{aligned} \quad (10)$$

The problem is then transformed by using $S_N = M^\top \tilde{S}_N M$ and $W(t) = M^\top \tilde{W} M$ in Eq.1.

The centralized solution is well-known (sampled data with zero-order hold) [6]:

$$U^j(t_i) = L_i^j [\hat{X}_i - X^R(t_i)] + U^{Rj}(t_i) \quad (11)$$

Here L_i^j is determined from

$$L_i^j = [V_d^j(t_{i+1}, t_i) + [B_d^j(t_{i+1}, t_i)]^\top S_{i+1} B_d^j(t_{i+1}, t_i)]^{-1} [B_d^j(t_{i+1}, t_i)]^\top S_{i+1} \Phi(t_{i+1}, t_i) \quad (12)$$

where

$$\dot{\Phi}(t, \tau) = A(t) \Phi(t, \tau); \quad \Phi(\tau, \tau) = I \quad (13)$$

$$B_d^j(t, t_i) = \int_{t_i}^t \Phi(t, \tau) B^j(\tau) d\tau \quad (14)$$

$$W_d(t_{i+1}, t_i) = \int_{t_i}^{t_{i+1}} \Phi(t, t_i)^\top W(t) \Phi(t, t_i) dt \quad (15)$$

$$V_d^j(t_{i+1}, t_i) = \int_{t_i}^{t_{i+1}} [B_d^j(t, t_i)^\top W(t) B_d^j(t, t_i) + V^j(t)] dt \quad (16)$$

$$A(t) = \left. \frac{\partial f}{\partial X} \right|_{X=X^R(t)} \quad B^j(t) = \left. \frac{\partial b^j}{\partial U^j} \right|_{U^j=U^{Rj}(t)} \quad (17)$$

and S_i is swept backward from S_N using

$$\begin{aligned} S_i = & \Phi(t_{i+1}, t_i)^\top S_{i+1} \Phi(t_{i+1}, t_i) + W_d(t_{i+1}, t_i) \\ & - \sum_{j=1}^K (L_i^j)^\top [V_d^j(t_{i+1}, t_i) + [B_d^j(t_{i+1}, t_i)]^\top S_{i+1} B_d^j(t_{i+1}, t_i)]^{-1} L_i^j \end{aligned} \quad (18)$$

The state estimate \hat{X}_i is generated by the “continuous/discrete” extended Kalman filter:

$$K_i = \bar{P}(t_i)H_i^\top (H_i\bar{P}(t_i)H_i^\top + R_i)^{-1}; \quad \bar{P}(t_1) = \bar{P}_1 \quad (19)$$

$$\hat{X}_i = \bar{X}(t_i) + K_i[Y_i - g(\bar{X}(t_i), t_i)]; \quad \bar{X}(t_1) = \bar{X}_1 \quad (20)$$

$$\hat{P}_i = (I - K_i H_i)\bar{P}(t_i)(I - K_i H_i)^\top + K_i R_i K_i^\top = \left[\bar{P}(t_i)^{-1} + \sum_{j=1}^K (H_i^j)^\top (R_i^j)^{-1} H_i^j \right]^{-1} \quad (21)$$

$$\dot{\bar{X}}(t) = f(\bar{X}, t) + \sum_{j=1}^K b^j(U^j, t); \quad \bar{X}(t_i) = \hat{X}_i \quad (22)$$

$$\bar{P}(t) = \Phi(t, t_i)\hat{P}_i\Phi(t, t_i)^\top + Q_d(t, t_i) \quad (23)$$

where $Y_i = [Y_i^1; Y_i^2; \dots, Y_i^K]$, $g = [g^1; g^2; \dots, g^K]$, $R_i = \text{diag}([R_i^1, R_i^2, \dots, R_i^K])$, and

$$H_i = \left. \frac{\partial g}{\partial \bar{X}} \right|_{\bar{X}=\bar{X}(t_i)} \quad (24)$$

$$Q_d(t, t_i) = \int_{t_i}^t \Phi(t, \tau) Q(\tau) \Phi(t, \tau)^\top d\tau \quad (25)$$

$$\mathbb{E}[\bar{X}_1 - X_1] = 0; \quad \mathbb{E}[(\bar{X}_1 - X_1)(\bar{X}_1 - X_1)^\top] = \bar{P}_1 \quad (26)$$

Problem Solution

As in [1], decompose the state space into a control-dependent partition and a data-dependent partition:

$$\begin{aligned} \hat{X}_i &= X^C(t_i) + \hat{x}_i^D \\ \bar{X}(t_i) &= X^C(t_i) + \bar{x}^D(t_i); \quad X^C(t_1) = \bar{X}_1, \quad \bar{x}^D(t_1) = 0 \end{aligned} \quad (27)$$

Define

$$\tilde{y}_i^j = Y_i^j - g^j(X^C(t_i), t_i) \quad (28)$$

then a local, linearized Kalman filter (LKF) operating only on the local data is

$$K_i^j = \bar{P}^j(t_i)[H_i^j]^\top (H_i^j \bar{P}^j(t_i)[H_i^j]^\top + R_i^j)^{-1}; \quad \bar{P}^j(t_1) = \bar{P}_1 \quad (29)$$

$$\hat{x}_i^{Dj} = \bar{x}^{Dj}(t_i) + K_i^j [\hat{y}_i^j - H_i^j \bar{x}^{Dj}(t_i)] \quad (30)$$

$$\hat{P}_i^j = (I - K_i^j H_i^j) \bar{P}^j(t_i) (I - K_i^j H_i^j)^\top + K_i^j R_i^j (K_i^j)^\top \quad (31)$$

$$\bar{x}^{Dj}(t) = \Phi(t, t_i) \hat{x}_i^{Dj} \quad (32)$$

$$\bar{P}^j(t) = \Phi(t, t_i) \hat{P}_i^j \Phi(t, t_i)^\top + Q_d(t, t_i) \quad (33)$$

where

$$H_i^j = \left. \frac{\partial g^j}{\partial X} \right|_{X=X^C(t_i) + \bar{x}^{Dj}(t_i)} \quad (34)$$

Note that Eqs. 29 - 31 can be iterated by evaluating Eq. 34 with the previous iteration's value for \hat{x}_i^{Dj} in place of $\bar{x}^{Dj}(t_i)$. In some applications, the accuracy of \hat{x}_i^{Dj} can thus be improved.

Now the algorithm of [1] can be utilized. This algorithm is based on writing \hat{x}_i^D as a linear combination of the local estimates \hat{x}_i^{Dj} and an additional data-dependent vector calculated locally:

$$\hat{x}_i^D = \sum_{j=1}^K \left[\hat{P}_i (\hat{P}_i^j)^{-1} \hat{x}_i^{Dj} + h_i^j \right] \quad (35)$$

where

$$h_i^j = F_i h_{i-1}^j + G_i^j \bar{x}^{Dj}(t_i); \quad h_1^j = 0 \quad (36)$$

$$F_i = \left[I - \sum_{j=1}^K \hat{P}_i (H_i^j)^\top (R_i^j)^{-1} H_i^j \right] \Phi(t_i, t_{i-1}) = \hat{P}_i \bar{P}_i^{-1} \Phi(t_i, t_{i-1}) \quad (37)$$

$$G_i^j = F_i \hat{P}_{i-1} (\bar{P}_{i-1}^j)^{-1} \Phi(t_i, t_{i-1})^{-1} - \hat{P}_i (\bar{P}_i^j)^{-1} \quad (38)$$

The essence of the algorithm of [1] is that one need not reconstruct the globally optimal state \hat{x}_i^D via Eq. 35 if one only needs to compute the globally optimal control. Instead, define

$$\alpha_i^{\ell j} = [B_d^\ell(t_{i+1}, t_i)]^\top S_{i+1} \Phi(t_{i+1}, t_i) \left[\hat{P}_i (\hat{P}_i^j)^{-1} \hat{x}_i^{Dj} + h_i^j \right] \quad (39)$$

Then, Eqs. 11 and 12 can be rewritten as

$$U^j(t_i) = [V_d^j(t_{i+1}, t_i) + [B_d^j(t_{i+1}, t_i)]^\top S_{i+1} B_d^j(t_{i+1}, t_i)]^{-1} \quad (40)$$

$$\times \left\{ [B_d^j(t_{i+1}, t_i)]^\top S_{i+1} \Phi(t_{i+1}, t_i) [\hat{X}_i^C - X^R(t_i)] + \sum_{\ell=1}^K \alpha_i^{j\ell} \right\} + U^{Rj}(t_i)$$

Note that even though the approach does not require a global Kalman filter, it does require the global covariance matrix \hat{P}_i in Eqs. 37, 38, and 39. Fortunately, Eqs. 21 and 23 do not require the measurement data, but Eq. 21 does require knowledge of the information contribution of each node, $(H_i^j)^\top (R_i^j)^{-1} H_i^j$. This consideration will be addressed below.

Chang [7] shows that at least Eqs. 36 and 38 can be avoided if instead of maintaining both \hat{x}_i^{Dj} and h_i^j independently, they are combined in a single vector, defined as follows:

$$\hat{\psi}_i^j = \hat{P}_i (\hat{P}_i^j)^{-1} \hat{x}_i^{Dj} + h_i^j \quad (41)$$

so that

$$\alpha_i^{j\ell} = [B_d^\ell(t_{i+1}, t_i)]^\top S_{i+1} \Phi(t_{i+1}, t_i) \hat{\psi}_i^j \quad (42)$$

Reference [7] shows that a recursion for $\hat{\psi}_i^j$ is

$$\hat{\psi}_i^j = F_i \bar{\psi}^j(t_i) + K_i^j \tilde{y}_i^j; \quad \bar{\psi}^j(t) = \Phi(t, t_i) \hat{\psi}_i^j, \quad \hat{\psi}_1^j = 0 \quad (43)$$

If there is no reason to maintain a locally optimal state estimate, then Eqs. 30 and 32 can also be avoided.

Remarks

Note that the nonlinear state and output relations, Eqs. 2 and 4, appear nowhere in Eqs. 35-41. Partitioning all of the initial condition information into X^C allows the local Kalman filter to directly operate only on x^{Dj} and hence remain linear.

In either the original, linear formulation or the current extended formulation, the control-dependent and data-dependent partitions of the state will tend to individually diverge from the truth as more data are incorporated and more controls are executed. This is not an issue for the linear problem, but in the present case, it could cause the linearizations to be compromised. Therefore, it is important that evaluations of partial derivatives such as Eqs. 34 and 17 be evaluated on either the desired trajectory X^R or the locally optimal whole state estimate, $X^C + \bar{x}^{Dj}$. An iterated update may also be desirable.

In certain applications, the control may only be active during a finite time interval, but the estimator may nevertheless continue to operate at all times. In such circumstances, it may be appropriate to revert to the EKF when the controller is inactive. Then, as each actuation epoch arrives, the LKF would be re-initialized using the latest EKF state estimate.

As mentioned previously, the local reconstruction of the globally optimal control requires that the covariance of the global state estimate, although not the global state estimate itself, be maintained at each node. This presents some difficulties. The measurement partial derivatives for the non-local nodes, H_i^ℓ , $\ell \neq j$, are required by Eq. 21. According to Eq. 34, the non-local data-dependent state partitions, $\bar{x}^{D\ell}$, are required, implying an additional inter-node communication requirement. However, as mentioned above, Eq. 34 can instead be evaluated on the desired trajectory X^R , which is the same at all nodes. This procedure will however deny the possibility of an iterated update. In fact, it could be argued that the globally optimal covariance should be computed using the globally optimal data-dependent state partition, \bar{x}^D , which is unavailable at any node. In practice, it may be adequate to merely compute H_i^ℓ , $\ell \neq j$ using local information, accepting the induced suboptimality in the reconstruction of the global covariance.

A more significant practical issue with the local reconstruction of the globally optimal covariance

is that the measurement updates may not occur at uniform intervals at all the nodes. In fact, if as is typical, the local Kalman filters employ an editing procedure, the covariance update interval becomes non-deterministic. These issues can be addressed by requiring the additional transmission of a *semaphore* from each node to all the other nodes every time a local measurement is successfully incorporated. The semaphore can be as simple as a single bit, which when positive, indicates that the global covariances at the other nodes should be updated to reflect that a measurement was processed by the node transmitting the semaphore.

In [2], reference [1] is significantly generalized, and it is shown that the local estimators need not share the state space of the implicit globally optimal centralized estimator, under the restriction that linear relationships among the elements of the measurement vector present in the global model are preserved in the local model. A restricted subset of such conditions is the case in which the local model's states are a subset of the global model's states. The present work applies only to this restricted case.

Conclusions

A straightforward, ad-hoc extension of [1] is proposed that allows its use for commonly encountered classes of problems that are currently solved with the extended Kalman filter. As such, the proposed approach shares many of the limitations of the EKF, such as a lack of guaranteed stability. Nevertheless, it can be expected that the wide successful usage of the EKF implies that the current approach will suffice for many problems of practical interest.

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